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AN IMPROVED VERSION OF REGRESSION RATIO ESTIMATOR WITH TWO AUXILIARY VARIABLES IN SAMPLE SURVEYS

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ABSTRACT

This paper considers a family of estimators of population mean $\overline{Y}$ of the variable *y* under study using information on two auxiliary variables *x* and *z* under simple random sampling without replacement (SRSWOR) scheme. The expressions for bias and mean-squared error of the proposed family of estimators are obtained up to the first degree of approximation. In addition to many, Mohanty (1967), Khare and Srivastava (1981) and Upadhyaya and Singh (1984) estimators are identified as particular members of the suggested family. Asymptotic optimum estimator (AOE) in the family is identified with its approximate mean-squared error expression.

Key words: Family of estimators, Study variate, Auxiliary variate, Bias, Mean-squared error.

1. Introduction

In sample surveys, it is a tradition to use auxiliary information for improving the precision of estimator(s) of the population parameter(s) of the variable under study. Out of many, ratio, regression and product methods of estimation are good examples in this context. The ratio method [Cochran (1940, 1942)] has been widely used when the correlation between the character under study '*y*' and the auxiliary character '*x*' is positive. If this correlation is negative, a product estimator envisaged by Robson (1957) and Murthy (1964) may be used instead of a ratio estimator. Hansen et al. (1953) suggested that the difference estimator

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which was subsequently modified to give the linear regression estimator is used when the regression line of y on x does not pass through the neighbourhood of the origin. In many survey situations information on more than single auxiliary variables is available to the investigators. Several authors including Olkin (1958), Desraj (1965), Shukla (1966), Singh (1967), John (1969), Srivastava (1971), Rao and Mudholkar (1967), Sahai et al. (1980), Singh and Upadhyaya (1981), Agrawal and Panda (1993, 94), Srivastava and Jhajj (1983), Mohanty and Pattanaik (1984), and Tuteja and Bahl (1991) have presented various extensions of the ratio and product estimators to the case where multiple auxiliary variables are used to increase the precision.

Let the population consist of N identifiable sampling units, the i-th unit labelled as U_i ($i=1,2, \ldots, N$). In the general case a simple random sample of size n of the form $(y_1, x_1), (y_2, x_2), \ldots, (y_n, x_n)$ is taken without replacement scheme from the finite population U. We are interested in estimating the population mean $\overline{Y}$ of y when the population mean $\overline{X}$ of x is known. The classical ratio and regression estimators for $\overline{Y}$ are respectively given by

$$\overline{y}_R = \overline{y}\frac{\overline{X}}{\overline{x}} \tag{1.1}$$

and

$$\overline{y}_{lr} = \overline{y} + b_{yx}(\overline{X} - \overline{x}) \tag{1.2}$$

where $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$, $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, and $b_{yx} = \frac{s_{yx}}{s_x^2}$ is the sample estimate of the population regression coefficient of y on x, $s_{yx} = \frac{1}{(n-1)}\sum_{i=1}^{n}(x_i - \overline{x})^2$, $s_x^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(x_i - \overline{x})^2$.

Further, suppose that the information on two auxiliary variables x and z is available. The 'z' may be the value at some previous time when a complete census was taken, and x is another auxiliary variable which we come across when y is being measured and it is highly correlated with y. For example, let us consider that we have to estimate the total yield of paddy in a state for which we have selected blocks as our sampling units. The area under cultivation for all blocks is known for a previous year (z_i). While measuring the total yield (y_i) for blocks selected in the sample we find that the yield has been affected by rain (x_i), which is highly correlated with the yield, x being known for all the blocks, see Mohanty (1967, pp. 16–17).

Assuming that $\bar{X}$ and $\bar{Z}$, the population means of auxiliary characters *x* and *z*, are known and there is high correlation between *y* and *x*, *y* and *z*, whereas there may or may not be any correlation between *x* and *z*, Mohanty (1967) suggested a regression ratio estimator for $\bar{Y}$ as

$$\bar{y}_{RR} = \bar{y}_{lr} \frac{\bar{Z}}{\bar{z}}$$

$$= [\bar{y} + b_{yx}(\bar{X} - \bar{x})] \frac{\bar{Z}}{\bar{z}} \tag{1.3}$$

where $\bar{z} = \frac{1}{n}\sum_{i=1}^{n} z_i$ is the sample mean of *z* based on *n* observations drawn by SRSWOR.

Motivated by Srivastava (1967), Khare and Srivastava (1981) suggested a generalization of $\bar{y}_{RR}$ at (1.3) for $\bar{Y}$ as

$$\bar{y}_{RR}^{(\alpha)} = \bar{y}_{lr} \left(\frac{\bar{z}}{\bar{Z}}\right)^{\alpha}$$

$$= [\bar{y} + b_{yx}(\bar{X} - \bar{x})] \left(\frac{\bar{z}}{\bar{Z}}\right)^{\alpha} \tag{1.4}$$

where α is a suitably chosen constant.

Further, following Srivenkataramana (1980), Upadhyaya and Singh (1984) suggested a dual to regression ratio estimator for $\bar{Y}$ as

$$\bar{y}_{RR}^{(d)} = \bar{y}_{lr} \frac{\bar{z}^*}{\bar{Z}}$$

$$= [\bar{y} + b_{yx}(\bar{X} - \bar{x})] \frac{\bar{z}^*}{\bar{Z}} \tag{1.5}$$

where $\bar{z}^* = \frac{(N\bar{Z} - n\bar{z})}{(N-n)}$.

Keeping the form of $\bar{y}_{RR}$, $\bar{y}_{RR}^{(\alpha)}$ and $\bar{y}_{RR}^{(d)}$ in view we define a family of estimators for $\bar{Y}$ as

$$\bar{y}_{RR}^{(h)} = \bar{y}_{lr}\ h(u) \tag{1.6}$$

where $u = \dfrac{\bar{z}}{\bar{Z}}$, $h(u)$ is a function of u such that $h(1) = 1$ satisfying the following regularity conditions [see, Srivastava (1971, p. 405)]:

(i) Whatever be the sample chosen, u assumes values in a bounded, closed convex subset S of one dimensional real space containing the point '1',
(ii) In S, the function $h(u)$ is continuous and bounded,
(iii) The first and second partial derivatives of $h(u)$ exist and are continuous and bounded in S.

It may easily be observed that the estimators $\bar{y}_{RR}$, $\bar{y}_{RR}^{(\alpha)}$ and $\bar{y}_{RR}^{(d)}$ and Walsh (1970), Reddy (1973, 1974) type estimator

$$\bar{y}_{RR}^{(1)} = \bar{y}_{lr}\ [1+\alpha(u-1)]^{-1},$$

Gupta (1978) type estimators

$$\bar{y}_{RR}^{(2)} = \bar{y}_{lr}\ u^{-1}\left[1+(1-\alpha)u^{-1}\right],$$

$$\bar{y}_{RR}^{(3)} = \bar{y}_{lr}\ u\left[\alpha+(1-\alpha)u\right],$$

Sahai (1979) type estimator

$$\bar{y}_{RR}^{(4)} = \bar{y}_{lr}\ \frac{[\alpha+(1-\alpha)u]}{[\alpha u+(1-\alpha)]},$$

Sahai and Ray (1980) type estimator

$$\bar{y}_{RR}^{(5)} = \bar{y}_{lr}\ \left[2-u^{\alpha}\right],$$

Vos (1980), Adhvaryu and Gupta (1983) type estimators

$$\bar{y}_{RR}^{(6)} = \bar{y}_{lr}\ \left[\alpha+(1-\alpha)u^{-1}\right],$$

$$\bar{y}_{RR}^{(7)} = \bar{y}_{lr}\ [\alpha+(1-\alpha)u],$$

Srivenkataramana and Tracy (1980, 1981) type estimators

$$\bar{y}_{RR}^{(8)} = \bar{y}_{lr}\ \left[(1+\theta^{*})-\theta^{*}u\right],$$

$$\bar{y}_{RR}^{(9)} = \bar{y}_{lr}\ \left[(1-\theta^{**})+\theta^{**}u\right],$$

Tripathi (1980), Chaubey et al. (1984) type estimator

$$\bar{y}_{RR}^{(10)} = \bar{y}_{lr}\left[(1-\alpha)+\alpha\, u\right],$$

Srivenkataramana (1980) type estimator

$$\bar{y}_{RR}^{(11)} = \bar{y}_{lr}\left[(1+g)-g\, u\right]^{-1},$$

Ray and Sahai (1980) type estimators

$$\bar{y}_{RR}^{(12)} = \bar{y}_{lr}\,\frac{(\alpha+\theta\, u)}{\left[u+(\alpha+\theta-1)\right]},$$

$$\bar{y}_{RR}^{(13)} = \bar{y}_{lr}\,\frac{(\alpha+\theta\, u)}{\left[1+(\alpha+\theta-1)u\right]},$$

Singh and Ruiz Espejo (2003) type estimator

$$\bar{y}_{RR}^{(14)} = \bar{y}_{lr}\left[\alpha\, u^{-1}+(1-\alpha)u\right],$$

Singh and Shukla (1987) type estimator

$$\bar{y}_{RR}^{(15)} = \bar{y}_{lr}\,\frac{\left[(\alpha-2)\left\{(\alpha-1)+(\alpha-3)(\alpha-4)\right\}+(n/N)(d-1)(d-4)\, u\right]}{\left[(\alpha-1)\left\{(\alpha-2)+(n/N)(\alpha-4)\right\}+(\alpha-2)(\alpha-3)(\alpha-4)\, u\right]},$$

Mohanty and Sahoo (1987) type estimators

$$\bar{y}_{RR}^{(16)} = \bar{y}_{lr}\,\frac{u}{\left[(1-\alpha)u+\alpha\right]},$$

$$\bar{y}_{RR}^{(17)} = \bar{y}_{lr}\left[(1-\alpha)u^{2}+\alpha\, u\right]^{-1},$$

Naik and Gupta (1991 a) type estimators

$$\bar{y}_{RR}^{(18)} = \bar{y}_{lr}\left[(1+\alpha)-\alpha\, u\right],$$

$$\bar{y}_{RR}^{(19)} = \bar{y}_{lr}\,\frac{u}{\left[1+(1+\alpha)(u-1)\right]},$$

Naik and Gupta (1991 b) type estimator

$$\bar{y}_{RR}^{(20)} = \bar{y}_{lr}\left[\frac{\left\{1+a(u-1)\right\}}{\left\{1+b(u-1)\right\}}\right]^{\alpha},$$

Diana (1993) type estimator

$$\bar{y}_{RR}^{(21)} = \bar{y}_{lr}\left[u^{\delta}\left\{a+(1-a)\,u^{\phi}\right\}^{\alpha}\right],$$

Sisodia and Dwivedi (1981) type estimator

$$\bar{y}_{RR}^{(22)} = \bar{y}_{lr}\left[b^{*}u+(1-b^{*})\right]^{-1},$$

Bansal and Singh (1992–93) type estimator

$$\bar{y}_{RR}^{(23)} = \bar{y}_{lr}\left[b^{*}u+(1-b^{*})\right]^{-\alpha},$$

Upadhyaya and Singh (1999) type estimators

$$\bar{y}_{RR}^{(24)} = \bar{y}_{lr}\left[c\,u+(1-c)\right]^{-1},$$

$$\bar{y}_{RR}^{(25)} = \bar{y}_{lr}\left[c_1\,u+(1-c_1)\right]^{-1},$$

$$\bar{y}_{RR}^{(26)} = \bar{y}_{lr}\left[c\,u+(1-c)\right],$$

$$\bar{y}_{RR}^{(27)} = \bar{y}_{lr}\left[c_1\,u+(1-c_1)\right],$$

Mohanty and Sahoo (1995) type estimators

$$\bar{y}_{RR}^{(28)} = \bar{y}_{lr}\left[d^{*}u+(1-d^{*})\right]^{-1},$$

$$\bar{y}_{RR}^{(29)} = \bar{y}_{lr}\left[d^{*}u+(1-d^{*})\right]^{-\alpha},$$

$$\bar{y}_{RR}^{(30)} = \bar{y}_{lr}\left[d_1\,u+(1-d_1)\right]^{-1},$$

$$\bar{y}_{RR}^{(31)} = \bar{y}_{lr}\left[d_1\,u+(1-d_1)\right]^{-\alpha},$$

etc. all being equal to $\bar{y}_{lr}$ at $\bar{z}=\bar{Z}$, belong to the proposed family $\bar{y}_{RR}^{(h)}$, where $b^{*}=\dfrac{\bar{Z}}{\bar{Z}+C_z}$, $c=\dfrac{\bar{Z}\beta_2(z)}{\bar{Z}\beta_2(z)+C_z}$, $c_1=\dfrac{\bar{Z}C_z}{\bar{Z}C_z+\beta_2(z)}$, $d^{*}=\dfrac{\bar{Z}}{\bar{Z}+z_m}$, $d_1=\dfrac{\bar{Z}}{\bar{Z}+z_M}$, $g=\dfrac{n}{(N-n)}$, $\theta^{*}=\dfrac{\bar{Z}}{(\alpha-\bar{Z})}$, $\theta^{**}=\dfrac{\bar{Z}}{(\alpha+\bar{Z})}$, $0\le\theta\le1$, C_z and $\beta_2(z)$ are known coefficients of variation and kurtosis of z, z_m and z_M are minimum and maximum z-values respectively , and $(\alpha, a, b, \delta, \phi)$ are suitably chosen constants.

2. Properties of the proposed family $\bar{y}_{RR}^{(h)}$

Under the usual assumptions, by Taylor's expansion method, it is observed that the bias of $\bar{y}_{RR}^{(h)}$ is of $o(n^{-1})$. Denoting the first order partial derivative of $h(u)$ with respect to u, at the point u=1, by $h_1(1)$, we obtain the variance of $\bar{y}_{RR}^{(h)}$ to $o(n^{-1})$, as

$$V(\bar{y}_{RR}^{(h)}) = V(\bar{y}_{lr}) + \frac{(N-n)}{n(N-1)} \bar{Y}^2 C_z^2 \left(h_1(1)\right)^2 \left[1 + 2\frac{C_y}{C_z}\left(\frac{\rho_{yz} - \rho_{yx}\rho_{xz}}{h_1(1)}\right)\right], \quad (2.1)$$

where

$$V(\bar{y}_{lr}) = \frac{(N-n)}{n(N-1)} \bar{Y}^2 C_y^2 (1-\rho_{yx}^2) \quad (2.2)$$

is the variance of usual linear regression estimator $\bar{y}_{lr}$ to $o(n^{-1})$, ρ_{yx}, ρ_{yz} and ρ_{xz} are the correlation coefficients between (y,x), (y,z) and (x,z) respectively, C_y and C_z are the coefficients of variation of y and z respectively.

The value of $h_1(1)$ that minimizes $V(\bar{y}_{RR}^{(h)})$ is

$$h_{1(\text{opt})} = -(\rho_{yz} - \rho_{yx}\rho_{xz})C_y / C_x \quad (2.3)$$

Thus the resulting (optimal) variance of $V\left(\bar{y}_{RR}^{(h)}\right)$ is given by

$$V(\bar{y}_{RR}^{(h)}) = V(\bar{y}_{lr}) - \frac{(N-n)}{n(N-1)} \bar{Y}^2 C_y^2 (\rho_{yz} - \rho_{yx}\rho_{xz})^2$$

$$= \frac{(N-n)}{n(N-1)} \bar{Y}^2 C_y^2 (1-\rho_{yx}^2)\left[1 - \frac{(\rho_{yz} - \rho_{yx}\rho_{xz})^2}{(1-\rho_{yx}^2)}\right] \quad (2.4)$$

Thus, we have the following theorem.

Theorem 2.1: Up to terms of order n^{-1},

$$V(\overline{y}_{RR}^{(h)}) \geq \frac{(N-n)}{n(N-1)}\overline{Y}^2 C_y^2(1-\rho_{yx}^2)\left[1-\frac{(\rho_{yz}-\rho_{yx}\rho_{xz})^2}{(1-\rho_{yx}^2)}\right]$$

with equality sign holding if

$$h_1(1) = (\rho_{yz}-\rho_{yx}\rho_{xz})(C_y / C_x).$$

It should be noted that any estimator attaining the variance (2.4) may be termed a minimum variance bound (MVB) estimator. Obviously, the MVB estimator for the family is not unique.

The variance of any estimator belonging to the proposed family $\overline{y}_{RR}^{(h)}$ can be obtained easily just by putting the proper value of $h_1(1)$ in (2.1). For example, putting $h_1(1) = -1$, α, $-g$ in (2.1) the variances of $\overline{y}_{RR}$, $\overline{y}_{RR}^{(\alpha)}$ and $\overline{y}_{RR}^{(d)}$ defined at (1.3), (1.4) and (1.5) respectively can be obtained.

We note that the family $\overline{y}_{RR}^{(h)}$ at (1.6) does not include even simple difference type estimators such as

$$\overline{y}_{RR(1)} = \overline{y}_{lr} - \alpha(u-1). \tag{2.5}$$

However, it is easily shown that if we consider a family of estimators wider than (1.6), defined by

$$\overline{y}_{RR}^{(H)} = H(\overline{y}_{lr}, u) \tag{2.6}$$

where $H(\overline{y}_{lr}, u)$ is a function of $\overline{y}_{lr}$ and u such that

$$\left.\begin{aligned} &H(\overline{y}_{lr}, u)\Big|_{(\overline{Y},\overline{X},1)} = H(\overline{Y}, 1) = \overline{Y} \\ &\Rightarrow H_1(\overline{Y},1) = \frac{\partial H(.)}{\partial \overline{y}_{lr}}\Bigg|_{(\overline{Y},\overline{X},1)} = 1 \end{aligned}\right\} \tag{2.7}$$

the minimum asymptotic variance of $\overline{y}_{RR}^{(H)}$ is equal to (2.4) and is not reduced. The estimator $\overline{y}_{RR(1)}$ is a member of the family (2.6) and attain the optimal (minimum) variance (2.4) for optimum value of α in $\bar{y}_{RR(1)}$, which can be obtained by (2.3).

3. Another family of estimators

As it is noted earlier that the information on variable z is available for all the units, thus letting $u = \frac{\bar{z}}{\bar{Z}}$, $v = \frac{s_z^2}{S_z^2}$, $s_z^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(z_i - \bar{Z})^2$, and $S_z^2 = \frac{\sum_{i=1}^{n}(z_i - \bar{Z})^2}{(N-1)}$, we define another family of estimators for $\bar{Y}$ as

$$\bar{y}_{RR}^{(g)} = \bar{y}_{lr}\ g(u,v) \tag{3.1}$$

where $g(u,v)$ is a function of (u,v) such that

$$g(1,1) = 1 \tag{3.2}$$

and it satisfies the following conditions [see, Srivastava and Jhajj (1981, p. 342)]:

1. The function $g(u,v)$ is continuous and bounded in Q.
2. The first and second order partial derivatives of $g(u,v)$ are continuous and bounded in Q.

Expanding the function $g(u,v)$ about the point (1,1) in a second order Taylor's series and noting that the second order partial derivatives are bounded, we have

$$\mathrm{E}\left(\bar{y}_{RR}^{(g)}\right) = \bar{Y} + 0(n^{-1})$$

and so the bias of $\bar{y}_{RR}^{(g)}$ is of the order n^{-1}.

We write

$C_y^2 = \frac{\mu_{020}}{\bar{Y}^2}$, $C_x^2 = \frac{\mu_{200}}{\bar{X}^2}$, $C_z^2 = \frac{\mu_{002}}{\bar{Z}^2}$, $\rho_{yx} = \frac{\mu_{110}}{\sqrt{\mu_{200}\mu_{020}}}$, $\rho_{yz} = \frac{\mu_{011}}{\sqrt{\mu_{020}\mu_{002}}}$,

$\rho_{xz} = \frac{\mu_{101}}{\sqrt{\mu_{020}\mu_{002}}}$, $\lambda_{102} = \frac{\mu_{102}}{\mu_{002}\sqrt{\mu_{200}}}$, $\lambda_{012} = \frac{\mu_{012}}{\mu_{002}\sqrt{\mu_{020}}}$, $\gamma_1(z) = \frac{\mu_{003}}{\mu_{002}^{3/2}}$,

$\beta_1(z)=\gamma_1^2(z)$, $\beta_2(z)=\dfrac{\mu_{004}}{\mu_{002}^2}$, $\lambda_{rst}=\dfrac{\mu_{rst}}{\left(\mu_{200}^{r/2}\ \mu_{020}^{s/2}\ \mu_{002}^{t/2}\right)}$,

$\mu_{rst}=\dfrac{\sum_{i=1}^{N}(x_i-\overline{X})^r(y_i-\overline{Y})^s(x_i-\overline{Z})^t}{N}$, (r,s,t) being non-negative integers.

To the first degree of approximation, the variance of $\overline{y}_{RR}^{(g)}$ is given by

$$V(\overline{y}_{RR}^{(g)})=V(\overline{y}_{lr})+\frac{(N-n)}{n(N-1)}\overline{Y}^2\Big[C_z^2g_1^2(1,1)+\{\beta_2(z)-1\}g_2^2(1,1)$$

$$+2\gamma_1(z)C_zg_1(1,1)g_2(1,1)+2C_yC_z(\rho_{yz}-\rho_{yx}\rho_{xz})g_1(1,1)$$

$$+2C_y(\lambda_{012}-\rho_{yx}\lambda_{102})g_2(1,1)\Big], \tag{3.3}$$

where $g_1(1,1)$ and $g_2(1,1)$ denote the first order partial derivatives of the function $g(u,v)$ with respect to u and v respectively about the point (1,1); $g_{(1)}(1,1)$ is given in (2.2).

The variance $V(\overline{y}_{RR}^{(g)})$ at (3.3) is minimized for

$$\left.\begin{aligned} g_{1(\mathrm{opt})}(1,1)&=\frac{\big[\gamma_1(z)(\lambda_{012}-\rho_{yx}\lambda_{102})-(\beta_2(z)-1)(\rho_{yz}-\rho_{yx}\rho_{xz})\big]C_y}{(\beta_2(z)-\beta_1(z)-1)C_z}\\ g_{2(\mathrm{opt})}(1,1)&=\frac{\big[\gamma_1(z)(\rho_{yz}-\rho_{yx}\rho_{xz})-(\lambda_{012}-\rho_{yx}\lambda_{102})\big]C_y}{(\beta_2(z)-\beta_1(z)-1)} \end{aligned}\right\} \tag{3.4}$$

Thus the resulting (optimal) variance of $\overline{y}_{RR}^{(g)}$ is given by

$$V_{\mathrm{opt}}(\overline{y}_{RR}^{(g)})$$

$$=\frac{(N-n)}{(N-1)n}\overline{Y}^2C_y^2(1-\rho_{yx}^2)\left[1-\frac{(\rho_{yz}-\rho_{yx}\rho_{xz})^2}{(1-\rho_{yx}^2)}\right.$$

$$\left.-\frac{D^2}{(1-\rho_{yx}^2)(\beta_2(z)-\beta_1(z)-1)}\right] \tag{3.5}$$

where $D = \left[\gamma_1(z)(\rho_{yz} - \rho_{yx}\rho_{xz}) - (\lambda_{012} - \rho_{yx}\lambda_{102})\right]$.

In (3.5), the first term on the right-hand side gives the approximate variance of the usual linear regression estimator $\bar{y}_{lr}$ in which $\bar{X}$ is used and the first two terms give the minimum (optimal) asymptotic variance of the family $\bar{y}_{RR}^{(h)}$ at (1.6) where both $\bar{X}$ and $\bar{Z}$ are used. The third term thus gives the reduction in asymptotic variance when S_z^2 is also used along with $\bar{X}$ and $\bar{Z}$.

Any parametric function $g(u,v)$ such that $g(1,1)=1$ and satisfies the conditions 1 and 2 can generate an estimator of the family $\bar{y}_{RR}^{(g)}$ at (3.1). The family of such estimators is very large. The following functions, for example, give some simple estimators of the family:

$$g(u,v) = u^{\alpha} v^{\beta},\ g(u,v) = w_1 u^{\alpha} + w_2 v^{\beta};\ w_1 + w_2 = 1,$$

$$g(u,v) = \left[1 + \alpha(u-1) + \beta(v-1)\right], g(u,v) = \frac{\left[1 + \alpha(u-1)\right]}{\left[1 + \beta(v-1)\right]},$$

$$g(u,v) = \left[1 - \alpha(u-1) - \beta(v-1)\right]^{-1} g(u,v) = \exp\left[\alpha(\mathrm{u}-1) + \beta(\mathrm{v}-1)\right].$$

The optimum values of α and β in the estimators defined by above functions are obtained from the conditions (3.4) and with these optimum values, the asymptotic minimum variance is given by (3.5).

We also state the following theorem:

Theorem 3.1: Up to terms of order n^{-1},

$$V(\bar{y}_{RR}^{(g)})$$

$$\geq \frac{(N-n)}{(N-1)n}\bar{Y}^2 C_y^2 (1-\rho_{yx}^2)\left[1 - \frac{(\rho_{yz} - \rho_{yx}\rho_{xz})^2}{(1-\rho_{yx}^2)} - \frac{D^2}{(1-\rho_{yx}^2)\left(\beta_2(z) - \beta_1(z) - 1\right)}\right]$$

with equality sign holding if

$$g_1(1,1) = g_{1(\mathrm{opt})}(1,1)$$
$$g_2(1,1) = g_{2(\mathrm{opt})}(1,1)$$

where $g_{1(\mathrm{opt})}(1,1)$ and $g_{2(\mathrm{opt})}(1,1)$ are defined in (3.4).

It can be easily shown that if we consider a wider family of estimators

$$\bar{y}_{RR}^{(G)} = G(\bar{y}_{lr}, u, v) \tag{3.6}$$

of $\bar{Y}$, where the function $G(.)$ satisfies $G(\bar{Y}, 1, 1) = \bar{Y}$ and $G_1(\bar{Y}, 1, 1) = 1$, $G_1(.)$ denoting the first partial derivative of $G(.)$ with respect to $\bar{y}_{lr}$, the optimal variance of $\bar{y}_{RR}^{(G)}$ is equal to (3.5) and is not reduced. The difference-type estimator

$$\bar{y}_{RR1}^{(G)} = \bar{y}_{lr} + \alpha(u-1) + \beta(v-1)$$

is a member of the family (3.6) but not that of the family defined in (3.1).

4. Efficiency Comparisons

In order to compare the estimators considered by Mohanty (1967), Khare and Srivastava (1981) as well as Singh and Upadhyaya (1984) with the proposed family of estimators, we write the variances of their estimators up to terms of order n^{-1} as

$$V(\bar{y}_{RR}) = V(\bar{y}_{lr}) + \frac{(N-n)}{n(N-1)}\bar{Y}^2 C_z^2 \left[1 - 2(\rho_{yz} - \rho_{yx}\rho_{xz})(C_y / C_z)\right] \tag{4.1}$$

$$V_{\text{opt}}(\bar{y}_{RR}^{(\alpha)}) = V_{\text{opt}}(\bar{y}_{RR}^{(h)}) \tag{4.2}$$

$$V_{\text{opt}}(\bar{y}_{RR}^{(d)}) = V(\bar{y}_{lr})$$

$$+ \frac{(N-n)}{n(N-1)}\bar{Y}^2 C_z^2 g^2 \left[1 - (2/g)(\rho_{yz} - \rho_{yx}\rho_{xz})(C_y / C_z)\right] \tag{4.3}$$

From (2.2), (2.4), (3.5), (4.1), (4.2) and (4.3), we have

$$V(\bar{y}_{lr}) - V_{\text{opt}}(\bar{y}_{RR}^{(h)}) = \frac{(N-n)}{n(N-1)}\bar{Y}^2 C_y^2 (\rho_{yz} - \rho_{yx}\rho_{xz})^2 \geq 0 \tag{4.4}$$

$$V(\bar{y}_{RR}) - V_{\text{opt}}(\bar{y}_{RR}^{(h)}) = \frac{(N-n)}{n(N-1)}\bar{Y}^2 \left[C_y(\rho_{yz} - \rho_{yx}\rho_{xz}) - C_z\right]^2 > 0 \tag{4.5}$$

$$V(\bar{y}_{RR}^{(d)}) - V_{\text{opt}}(\bar{y}_{RR}^{(h)}) = \frac{(N-n)}{n(N-1)}\bar{Y}^2 \left[C_y(\rho_{yz} - \rho_{yx}\rho_{xz}) - gC_z\right]^2 > 0 \tag{4.6}$$

$$V(\overline{y}_{RR}^{(h)}) - V_{\text{opt}}(\overline{y}_{RR}^{(g)}) = \frac{(N-n)}{n(N-1)}\overline{Y}^2\ C_y^2\ \frac{D^2}{\left(\beta_2(z)-\beta_1(z)-1\right)} > 0 \tag{4.7}$$

From (4.4) to (4.7), we have the following inequalities:

$$V_{\text{opt}}(\overline{y}_{RR}^{(g)}) < V_{\text{opt}}(\overline{y}_{RR}^{(h)}) < V(\overline{y}_{lr}) \tag{4.8}$$

$$V_{\text{opt}}(\overline{y}_{RR}^{(g)}) < V_{\text{opt}}(\overline{y}_{RR}^{(h)}) < V(\overline{y}_{RR}) \tag{4.9}$$

and

$$V_{\text{opt}}(\overline{y}_{RR}^{(g)}) < V_{\text{opt}}(\overline{y}_{RR}^{(h)}) < V(\overline{y}_{RR}^{(d)}) \tag{4.10}$$

Thus, we find that the proposed family $\overline{y}_{RR}^{(g)}$ at (3.1) is the best (in the sense of having least variance) among the estimators discussed here at its optimum conditions.

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